# A HODOGRAPH TRANSFORMATION WHICH APPLIES TO THE HEAVENLY EQUATION*

Manuel Mañas[1,†] and Luis Martínez Alonso[1,‡]
[1] *Departamento de Física Teórica II, Universidad Complutense*
*E28040 Madrid, Spain*
[†] `manuel@darboux.fis.ucm.es`
[‡] `luism@fis.ucm.es`

**Abstract**

A hodograph transformation for a wide family of multidimensional nonlinear partial differential equations is presented. It is used to derive solutions of the heavenly equation (dispersionless Toda equation) as well as a family of explicit ultra-hyperbolic selfdual vacuum spaces admiting only one Killing vector which is not selfdual, we also give the corresponding explicit Einstein–Weyl structures.

*Key words:* Heavenly equation, hodograph transformations, Einstein–Weyl spaces.
*1991 MSC:* 58B20.

*Partially supported by CICYT proyecto PB98–0821

# 1 Introduction

This work introduces a hodograph method to construct solutions of a ample family of nonlinear partial differential equations (PDE) among which we have the dispersionless Kadomtsev-Petviashivili (dKP) equation and the heavenly equation, relevant in the finding of Einstein–Weyl 3D spaces and selfdual vacuum Einstein spaces [1, 4, 16]. Hodograph transformations goes back the XIX century and as was shown by Riemann they are relevant in the discussion of hydrodynamic type systems, this hodograph transformation was generalized recently by Tsarev [15].

The layout of this paper is as follows. The next section is devoted to describe our scheme. Using solutions to a implicit relation we find solutions to nonlinear PDEs. Finally, in the §3 we apply these results to the finding of new solutions of the heavenly equation and of Einstein–Weyl spaces and the corresponding ultra-hyperbolic self-dual vacuum Einstein spaces. At this point is important to mention that our solutions are different from those found in [16].

# 2 The hodograph transformation

Our method begun with the following implicit equation for determining a scalar function $u = u(\boldsymbol{x})$ depending on $n$ variables $\boldsymbol{x} = (x_1, \dots, x_n)$

$$X_0(u) + \sum_{i=1}^{n} x_i \, X_i(u) = 0, \tag{1}$$

where $X_i$, $i = 0, 1, \dots, n$, are given functions of $u$. By denoting $x = x_1,\ t_i = x_{i+1},\ (i = 1, \dots, n-1)$, it follows that (1) is a hodograph transformation for the family of one-dimensional hydrodynamical systems

$$u_{t_i} = C_i(u) u_x, \quad i = 1, \dots, n-1, \tag{2}$$

where

$$C_i(u) := \frac{X_{i+1}(u)}{X_1(u)}. \tag{3}$$

Our main observation is that (1) provides solutions for the family of nonlinear PDEs

$$\sum_{|\alpha|=m} c_\alpha D^\alpha \phi = D^\beta F(\phi), \quad |\beta| = m, \tag{4}$$

where $D^\alpha$ and $D^\beta$ denote partial differentiation operations of a given order $m$ corresponding to $n$-component multi-indices $\alpha, \beta \in \mathbb{N}^n$, $F = F(\phi)$ is an arbitrary function and $c_\alpha$ are arbitrary constants. We are going to prove that a solution of (4) is given by the function

$$\phi(\boldsymbol{x}) := G(Q(u)),\ Q(u) := \frac{\sum_{|\alpha|=m} c_\alpha X^\alpha(u)}{X^\beta(u)}, \quad X^\sigma := X_{\sigma_1} \cdots X_{\sigma_n}, \tag{5}$$

where $G := (F_u)^{-1}$ is the inverse function of the derivative $F_u$ of $F$ with respect to $u$.

From (5) we deduce that

$$\begin{aligned}\phi_{x_i} &= G'(Q(u))Q'(u)u_{x_i} = G'(Q(u))Q'(u)\frac{X_i(u)}{X_j(u)}u_{x_j}\\ &= \frac{\partial}{\partial x_j}\int^u G'(Q(u))Q'(u)\frac{X_i(u)}{X_j(u)}\,\mathrm{d}\,u\end{aligned}$$

and therefore

$$D^\alpha\phi = D^\beta\int^u G'(Q(u))Q'(u)\frac{X^\alpha(u)}{X^\beta(u)}\,\mathrm{d}\,u.$$

From this relation we conclude

$$\begin{aligned}\sum_{|\alpha|=m} c_\alpha D^\alpha\phi &= D^\beta\int^u G'(Q(u))Q'(u)\frac{\sum_{|\alpha|=m} c_\alpha X^\alpha(u)}{X^\beta(u)}\,\mathrm{d}\,u\\ &= D^\beta\int^u G'(Q(u))Q'(u)Q(u)\,\mathrm{d}\,u,\end{aligned}$$

Now, if $H := F \circ G$ then

$$(H)'(Q) = (F' \circ G)(Q)G'(Q) = QG'(Q)$$

and hence

$$\sum_{|\alpha|=m} c_\alpha D^\alpha\phi = D^\beta\int^u H'(Q(u))Q'(u)\,\mathrm{d}\,u = D^\beta H(Q) = D^\beta F(\phi).$$

### Observations

1. In spite of the implicit nature of the hodograph relation (1) we can easily find explicit examples. We shall point out two of such cases:

- Assuming that

$$X_i(u) = \sum_{j=0}^{N} a_{ik} u^k, \quad i = 0, 1, \ldots, n,$$

(1) reads as

$$\sum_{k=0}^{N} A_k u^k = 0, \quad A_k = a_{0k} + \sum_{i=1}^{n} a_{ik} x_i,$$

and we will have $N$ complex roots

$$u_l = u_l(A_0, \ldots, A_N), \quad l = 1, \ldots, N.$$

For each of these roots we can evaluate $X_i(u_l)$ and get a family of solutions. As we know if $N \leq 4$ the roots can be gotten explicitly and therefore we will have an explicit algebraic function depending on the parameters $\{a_{ik}\}_{\substack{k=0,\ldots,4,\\ i=1,\ldots,n}}$.

- Another example appears by considering the Lambert function $W(z)$ which solves

$$W \exp(W) = z$$

and has been studied with certain detail [5]. The point here is that the implicit *fundamental* relation

$$a + bu + c\exp(u) = 0$$

is solved in terms of the Lambert function as follows

$$u = -W\Big(\frac{c}{b}\exp\Big(-\frac{a}{b}\Big)\Big) - \frac{a}{b}.$$

Thus, taking

$$X_i(u) = a_i + b_i u + c_i \exp(u),$$

with $a_i, b_i$ and $c_i$ arbitrary constants, the hodograph relation is

$$A + Bu + C\exp(u) = 0,$$

with

$$A := a_0 + \sum_{i=1}^{n} a_i x_i,\ B := b_0 + \sum_{i=1}^{n} b_i x_i,\ C := c_0 + \sum_{i=1}^{n} c_i x_i,$$

and the solution is

$$u = -W\Big(\frac{C}{B}\exp\Big(-\frac{A}{B}\Big)\Big) - \frac{A}{B}.$$

Hence, introducing the rational functions

$$r(\boldsymbol{x}) := \frac{a_0 + \sum_{i=1}^n a_i x_i}{b_0 + \sum_{i=1}^n b_i x_i}, \quad s(\boldsymbol{x}) := \frac{b_0 + \sum_{i=1}^n b_i x_i}{c_0 + \sum_{i=1}^n c_i x_i}$$

we can evaluate

$$X_i(\boldsymbol{x}) = a_i - b_i r(\boldsymbol{x}) - \big(b_i C - c_i s(\boldsymbol{x})\big) W\Big(\frac{1}{s(\boldsymbol{x})\exp r(\boldsymbol{x})}\Big)$$

and using (5) get a solution to the nonlinear PDE (4) in terms of the Lambert function.

2. We can employ the freedom in the the choice for the functions $\{X_i(u)\}_{i=1}^n$ to generate solutions of more general equations. Suppose a functional dependence of the form

$$\phi = (F'_\alpha)^{-1}(Q_\alpha) = (F'_\beta)^{-1}(Q_\beta),$$

for all $\alpha, \beta \in \mathcal{I}$, being $\mathcal{I}$ a set of $r = \operatorname{card}\mathcal{I}$ multi-indices of order $m$, and

$$Q_\gamma X^\gamma = \sum_{|\delta|=m} a_\delta X^\delta$$

Then, $\phi$ satisfies

$$\sum_{|\delta|=m} a_\delta D^\delta \phi = \frac{1}{r}\sum_{\gamma\in\mathcal{I}} D^\gamma F_\gamma(\phi).$$

For example, the hodograph relation

$$tT(u) + xX(u) + yY(u) = H(u)$$

provides solutions to

$$\frac{1}{2}(\phi_{xx} + \phi_{yy}) = (\exp(\phi))_{tt}, \quad \phi = \log\frac{X^2+Y^2}{2T^2}$$

as well to

$$\frac{1}{2}(\phi_{xx}+\phi_{yy})=(\exp(2\phi))_{xt},\quad \phi=\log\sqrt{\frac{X^2+Y^2}{4XT}}.$$

Thus, we need to fulfill

$$X^3+Y^2X=T^3.$$

So that, the solutions of

$$t\sqrt[3]{X(u)+Y^2(u)/X(u)}+xX(u)+yY(u)=H(u)$$

gives

$$\phi_{xx}+\phi_{yy}=(\exp(\phi))_{tt}+(\exp(2\phi))_{xt},\quad \phi=\frac{1}{3}\log(1+(Y/X)^2)-\log(2).$$

# 3 Applications in General Relativity

Among the nonlinear PDEs of the form for which our hodograph technique is applicable one finds an integrable equation: the dKP equation

$$\phi_{tx}+\phi_{yy}=(\phi^2)_{xx}.$$

This equation is relevant in hydrodynamics and our hodograph solutions were already discussed by Kodama in [7], the dKP equation appears in the construction of three-dimensional Einstein–Weyl spaces [4]. Another integrable equation within our family of PDEs is known with different names: heavenly equation, Boyer–Finley equation, dispersionless Toda and SU($\infty$)-Toda equation:

$$\phi_{z\bar{z}}+\kappa(\mathrm{e}^{\phi})_{tt}=0,\quad \kappa=\pm 1. \tag{6}$$

where $z=x+\mathrm{i}\,y$ and $\bar{z}=x-\mathrm{i}\,y$, $x,y,t,\phi\in\mathbb{R}$. This equation has been found to characterize self-dual vacuum Einstein spaces —of signature $(++--)$ (ultra-hyperbolic) for $\kappa=-1$ and $(++++)$ (Euclidean) when $\kappa=1$— having a non-selfdual Killing vector [1], while those having a selfdual Killing vector appear to be related to the wave (or Laplace) equation and the metrics are of Gibbons–Hawking type [6].

Very few solutions of the heavenly equation have been found. In first place a separation of variables $\phi(z,\bar{z},t)=\log(f(t))+\Phi(z,\bar{z})$ leads to the Liouville

equation [9] $\Phi_{z\bar{z}} = \mathrm{e}^{\Phi}$, whose general solution is well known. If one imposes a symmetry, say $z = \bar{z}$, then the equation linearize, after a hodographic change of variable [16] and in this form implicit solutions are gotten. Also in [14] an implicit solution based on the Painleve equations was given. In [3] a new explicit solution was presented, see also [10, 11]. Further studies of the geometry associated with the equation can be found in for example [2]. See also [13, 12] for further information regarding this equation.

The heavenly equation is also known as the dispersionless Toda equation and appears as an example of the so called Whitham hierarchies. It has been applied to the study of conformal transformations and topological field theory [8].

Our scheme provides solutions to the ultra-hyperbolic heavenly equation. The problem is to find solutions of the heavenly equation so that the corresponding metric does not have an additional Killing vector. Hence, following [11] the solutions of the heavenly equation must be non-invariant [10] (being the symmetry group composed of translations, scaling and conformal transformations), as these symmetries will carry to corresponding additional Killing vectors. This construction is equivalent to self-dual hyper-Kähler spaces and, as was shown by Ward [16] the heavenly equation can be used to generate Einstein–Weyl spaces in 3D.

To check that our scheme gives solutions of non-invariant type, for the ultra-hyperbolic case, we shall use the hodograph equation in the following form

$$t + \rho\,\mathrm{e}^{-\mathrm{i}\,\alpha(\rho)}\,z + \rho\,\mathrm{e}^{\mathrm{i}\,\alpha(\rho)}\,\bar{z} = h(\rho),$$

where $\alpha$ and $h$ are arbitrary functions of $u = \rho$ and the solution of the heavenly equation is given by

$$\phi = \log(\rho^2),$$

this form of the hodograph equation ensures that $\phi$ takes real values. Using polar coordinates $z = r\,\mathrm{e}^{\mathrm{i}\,\theta}$ we get the following hodograph relation

$$t + 2\rho r\cos(\alpha(\rho) - \theta) = h(\rho). \tag{7}$$

Now, following [10] we must check whether or not is possible to find constants $\alpha$ and $\beta$ and functions $a(z)$ and $b(\bar{z})$ such that the following equation holds

$$(\alpha + \beta t)\phi_t + a(z)\phi_z + b(\bar{z})\phi_{\bar{z}} = 2\beta - a'(z) - b'(\bar{z}). \tag{8}$$

Now, recalling that the hodograph relation implies

$$\rho_z = \frac{\rho\,\mathrm{e}^{-\mathrm{i}\,\alpha}}{D}, \quad \rho_{\bar z} = \frac{\rho\,\mathrm{e}^{\mathrm{i}\,\alpha}}{D}, \quad \rho_t = \frac{1}{D},$$

with

$$D := h' - (1 - \mathrm{i}\,\rho\alpha')\,\mathrm{e}^{-\mathrm{i}\,\alpha}\,z - (1 + \mathrm{i}\,\rho\alpha')\,\mathrm{e}^{\mathrm{i}\,\alpha}\,\bar z,$$

and introducing the notation

$$A(z) := a(z) - \beta z, \quad B(\bar z) := b(\bar z) - \beta \bar z$$

we can write (8) in the following form

$$\alpha + \beta h + A\rho\,\mathrm{e}^{-\mathrm{i}\,\alpha} + B\rho\,\mathrm{e}^{\mathrm{i}\,\alpha} = -(A' + B')F, \tag{9}$$

with

$$F := \frac{\rho D}{2}.$$

Now, if the functions $\{1, \rho\,\mathrm{e}^{-\mathrm{i}\,\alpha}, \rho\,\mathrm{e}^{\mathrm{i}\,\alpha}, h\}$ are linearly dependent,

$$\lambda_1 1 + \lambda_2\rho\,\mathrm{e}^{-\mathrm{i}\,\alpha} + \lambda_3\rho\,\mathrm{e}^{\mathrm{i}\,\alpha} + \lambda_4 h = 0, \tag{10}$$

for some constants $\lambda_i$, $i = 1, 2, 3, 4$, then (9) will be identically satisfied if $\alpha = \lambda_1, \beta = \lambda_2, A = \lambda_3$ and $B = \lambda_4$. The invariant solutions should appear also if (9) holds taking $x, y, t$ and $u$ as independent variables. In doing so must impose $A = A_1 z + A_0$ and $B = B_1\bar z + B_0$ together with the equations

$$\begin{aligned} A_1 - \frac{1}{2}(1 - \mathrm{i}\,\rho\alpha')(A_1 + B_1) &= 0, \\ B_1 - \frac{1}{2}(1 + \mathrm{i}\,\rho\alpha')(A_1 + B_1) &= 0, \\ \alpha + \beta h + A_0\rho\,\mathrm{e}^{-\mathrm{i}\,\alpha} + B_0\rho\,\mathrm{e}^{\mathrm{i}\,\alpha} + \frac{1}{2}(A_1 + B_1)\rho h' &= 0. \end{aligned}$$

The two first are equivalent to the ODE

$$\alpha' = \mathrm{i}\,\frac{A_1 - B_1}{A_1 + B_1}\frac{1}{\rho}$$

that implies

$$\alpha = \mathrm{i}\,\gamma\log\rho + C \Rightarrow \mathrm{e}^{-\mathrm{i}\,\alpha} = c\rho^{\gamma}, \quad \gamma := \frac{A_1 - B_1}{A_1 + B_1}. \tag{11}$$

while the third determines $h$ as a solution of the following ODE

$$\alpha + \beta h + A_0 c\rho^{1+\gamma} + B_0 c^{-1}\rho^{1-\gamma} + \frac{1}{2}(A_1 + B_1)\rho h' = 0,$$

whose solution is

$$h(\rho) = C\rho^{2\frac{\beta}{A_1+B_1}} - \frac{\alpha}{\beta} - \frac{cA_0}{A_1+B_1}\rho^{2\frac{A_1}{A_1+B_1}} - \frac{B_0}{c(A_1+B_1)}\rho^{2\frac{B_1}{A_1+B_1}}. \tag{12}$$

Generically, if neither (10) nor (11)-(12) hold it would be difficult to have an invariant solution. Introducing the notation $f_\pm(\rho) := \rho\,\mathrm{e}^{\mp\mathrm{i}\,\alpha}$, $F_\pm := f'_\pm/F'$ and $H := h'/F'$ taking taking $t$-derivatives of (9) we get

$$\beta H + AF_+ + BF_- = -(A' + B'), \tag{13}$$

$$\beta H^{(n)} + AF_+^{(n)} + BF_-^{(n)} = 0, \quad n \geq 1. \tag{14}$$

Thus, in order to have invariant solutions we must impose

$$\begin{vmatrix} H^{(n_1)} & F_+^{(n_1)} & F_-^{(n_1)} \\ H^{(n_2)} & F_+^{(n_2)} & F_-^{(n_2)} \\ H^{(n_3)} & F_+^{(n_3)} & F_-^{(n_3)} \end{vmatrix} = 0, \text{with } 0 < n_1 < n_2 < n_3, \quad n_j \in \mathbb{N}.$$

and therefore an infinite set of equations need to be satisfied by the solution $\rho(z, \bar{z}, t)$ of the hodograph relation.

Following [16] we know that any solution of $\phi$ of (6) defines an Einstein–Weyl space given by

$$\mathrm{d}\,l^2 = \mathrm{d}\,t^2 - 4\rho^2(\mathrm{d}\,r^2 + r^2\,\mathrm{d}\,\theta^2), \quad \omega = 2\phi_t\,\mathrm{d}\,t. \tag{15}$$

Thus, by introducing the change of variables $(t, r, \theta) \to (\rho, r, \psi)$

$$\begin{aligned} t &= h(\rho) - 2r\rho\cos\psi, \\ r &= r, \\ \theta &= \psi + \alpha(\rho), \end{aligned} \tag{16}$$

the corresponding Einstein–Weyl structure becomes explicitly given in terms of two arbitrary functions $\alpha(\rho)$ and $h(\rho)$ by

$$\begin{aligned} \mathrm{d}\,l^2 = {} & [(h' - 2r\cos\psi)^2 - 4r^2\rho^2(\alpha')^2]\,\mathrm{d}\,\rho^2 - 4\rho^2\sin^2\psi\,\mathrm{d}\,r^2 - 4r^2\rho^2\cos^2\psi\,\mathrm{d}\,\psi^2 \\ & - 2\rho\cos(\psi)(h' - 2r\cos\psi)\,\mathrm{d}\,\rho\,\mathrm{d}\,r - 4r\rho[(h' - 2r\cos\psi)\sin(\psi) - 2r\rho\alpha']\,\mathrm{d}\,\rho\,\mathrm{d}\,\psi \\ & - 2r\rho^2\sin 2\psi\,\mathrm{d}\,r\,\mathrm{d}\,\psi, \end{aligned}$$

and

$$\omega = \frac{4}{\rho}\frac{(h' - 2r\cos\psi)\,\mathrm{d}\,\rho - 2\rho\cos\psi\,\mathrm{d}\,r + 2r\rho\sin\psi\,\mathrm{d}\,\psi}{h' - 2r\cos\psi - 2r\rho\alpha'\sin\psi}\,\mathrm{d}\,t. \tag{17}$$

It should be noticed that (16)–(17) define a family of Einstein–Weyl structures different from that characterized by Ward in [16]. Indeed, Ward uses an hodograph transformation for determining all solutions of (6) independent on one of the spatial variables $x$ or $y$.

The corresponding ultra-hyperbolic vacuum Einstein metric in 4D is given by [11]

$$\begin{aligned}\mathrm{d}\,s^2 &= \phi_t\,\mathrm{d}\,l^2 - \frac{1}{\phi_t}[\mathrm{d}\,\tilde{t} + \mathrm{i}(\phi_z\,\mathrm{d}\,z - \phi_{\bar{z}}\,\mathrm{d}\,\bar{z})]^2 \\ &= \frac{2}{\rho D}\,\mathrm{d}\,l^2 - \frac{\rho D}{2}\big(\,\mathrm{d}\,\tilde{t} - \frac{4}{D}(\sin\psi\,\mathrm{d}\,r + r\cos\psi\,\mathrm{d}\,\psi + \alpha' r\cos\psi\,\mathrm{d}\,\rho)\big)^2,\end{aligned}$$

with $D = h' - 2r\cos\psi - 2r\rho\alpha'\sin\psi$.

# 4 Acknowledgements

The authors would like to acknowledge discussions with Alexander Mikhailov, Ian Strachan and Sergei Tsarev.

# References

[1] C. P. Boyer & J. D. Finley, J. Math. Phys. **23** (1982) 1126.

[2] D. M. J. Calderbank, Geom. Phys. **36** (2000) 152.

[3] D. M. J. Calderbank & K. P. Tod, Diff. Geom. Appl. **14** (2001) 199.

[4] M. Dujnaski, L. J. Mason & K. P. Tod, J. Geom. Phys. **37** (2001) 63.

[5] R. M. Corless, G. H. Gonnet, G. H. Hare, De. E. G. Jeffrey and D. E. Knuth, Adv. Comp. Math. **5** (1996) 329.

[6] G. W. Gibbons & S. W. Hawking, Phys. Lett. **B78** (1978) 430.

[7] Y. Kodama, Prog. Theor. Phys. Supppl. **95** (1988) 184.

[8] I.M.Krichever, Commun. Pure. Appl. Math. **47**, (1992) 437, Commun. Math. Phys. **143**, (1992) 415. M.Mineev-Weinstein, P.B.Wiegmann and A.Zabrodin, Phys. Rev. Lett. **84** (2000) 5106. J.Gibbons and S.P.Tsarev, Phys. Lett. **211A** (1996) 19, Phys. Lett. **258A** (1999) 263.

[9] J. Liouville, Journ. Math. Pure Appl. **18** (1853) 71.

[10] L. Martina, M. B. Sheftel & P. Winternitz, J. Phys. A: Math. Gen. **34** (2001) 9243.

[11] Y. Nutku & M. B. Sheftel, `arXiv:gr-qc/015088`.

[12] I. A. B. Strachan, J. Phys. A: Math. & Gen. **29** (1996) 6117.

[13] M. V. Saveliev, Theor.Math. Phys. **92** (1992) 456.

[14] K. P. Tod, Class. Quantum Grav.

[15] S. P. Tsarev, Sov. Math. Dokl. **31** (1985) 488. **12** (1995) 1535.

[16] R.S. Ward, Class. Quantum Grav. **7** (1990) L95.